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SOME FOUNDATIONS FOR EMPIRICAL STUDY IN THE EUCLIDEAN SPATIAL MODEL OF SOCIAL CHOICE

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The new results are: the asymptotic consistency of the sample yolk center and epsilon-core; NP-completeness of the 1-dimensional spatial location recovery system; a modification of the Poole-Rosenthal heuristic for multidimensional recovery; and fast algorithms to compute Simpson-Cramer points and supermajority win sets in fixed dimension.

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Some Foundations for Empirical Study in the Euclidean Spatial Model of Social Choice*

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Abstract

Recent results are surveyed, and some new results are given, that contribute towards a theoretical and computational basis for empirical study in the Euclidean spatial model. The results are of two types: asymptotic statistical consistency of sample estimators, and algorithmic methods for recovering spatial locations and computing various solution concepts.

The new results are: the asymptotic consistency of the sample yolk center and epsilon-core; NP-completeness of the 1-dimensional spatial location recovery problem; a modification of the Poole-Rosenthal heuristic for multidimensional recovery; and fast algorithms to compute Simpson-Cramer points and supermajority win sets in fixed dimension.

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1 Introduction

Experiments and historical data analysis with the spatial model of voting call for underpinning theoretical work. Two categories of results needed to support empirical research, that have only recently begun to receive attention, are computational methods and statistical convergence.

This paper surveys results of these types, contributes some new results, and states open problems for further research. We can attempt to give a fairly complete picture of the current status of this research area, on account of its newness. We hope that the work surveyed here will aid in the empirical study of the spatial model, and stimulate additional advances and applications. In the remainder of the introduction we explain the motivation for each of the two categories of results, and state the principal new results given here.

Computational methods. A typical scenario that requires a computational method arises as follows: a researcher has numerical data giving the locations of voter ideal points and of the resulting group choice. The researcher wants to determine whether the outcome lies in a particular solution set, to test the predictive power of the solution concept. This requires that the solution set be computed, with respect to the numerical data.

If the number of voters is small this computation can ordinarily be performed by hand. But as the number of voters grows, the difficulty of the computation may increase enormously.

We temporarily depart from our scenario to introduce informally some notions of computational complexity. In the field of computer science, computational requirements are generally measured as a function of the size of the problem (in this case, the number of voters) (see [15]). Larger size problems require more computer time: the key question is, at what rate of increase? Roughly speaking, an algorithm, or computational procedure, is considered "fast" if it requires computer time growing as a polynomial function of the problem size. An algorithm is considered "slow" if it requires time growing as an exponential function of the problem size. For example, a procedure that examined all pairs of voters would require time quadratic in problem size, and would be fast; a procedure that examined all subsets of voters would require about 2^n time, where n = the number of voters, and would be slow. There is a corresponding taxonomy of problems:

a problem is considered "easy" if there is a fast algorithm known to solve it; a problem is considered "hard" if no fast algorithm is known that solves it. The "NP-complete" problems are a prominent class of "hard" problems.

Returning to our scenario, we have the unfortunate situation that almost any solution concept worth its salt is "hard". This is a consequence of an observation and a theorem. The observation is, most good solution concepts coincide precisely with the Condorcet winner, when the latter exists. The theorem is, it is hard to determine if there is a Condorcet winner [1]. Therefore, most good solution concepts implicitly check if a Condorcet winner exists, and so computing them must be hard as well.

This does not mean that the researcher in our scenario is doomed to failure. But this does mean that our scenario demands genuine expertise in computational methods. As researchers begin to work with data from committees comprised of more than a handful of voters, (e.g. legislative assemblies), we must pay some serious attention to computational methods to support this work. Good algorithmic tools are needed to compute win sets and critical levels for various solution concepts, and to fit spatial locations to voting data[1].

The principal new results in the area of computational methods are: NP-completeness of one-dimensional recovery of spatial locations from historical voting data; improvements in heuristics for multidimensional spatial location recovery; fast algorithms to compute Simpson-Kramer points and supermajority win sets in fixed (low) dimension; fast algorithms to compute membership and critical levels of epsilon-cores, in 2 dimensions.

Statistical consistency.

Empirical studies with the spatial model will almost inevitably involve some degree of randomness, due to variations in individual behavior, imprecision in measurements, use of sample data (e.g. survey or poll data), and other uncertainties of information. As discussed in [29], we would like

[1] Fast algorithmic tools are a practical possibility, despite the theoretical "hardness" just explained. The whole field of heuristics in operations research is devoted to practical means of overcoming computational difficulty. One reason this is possible is that the taxonomy given is "worst-case" – many or most numerical cases of a problem may be easy to solve, even if the problem is hard. For spatial data in particular, the theorem cited ([1]) is true only when the dimension is permitted to be large. If the dimension is fixed at a small level (e.g. 2 in most current empirical studies) the problem, though not trivial, is technically "easy".

to extract data on the ideal points of committee members or a population, and make a prediction regarding the outcome based on a solution concept. Can we be confident that a prediction based on polls taken one day will be close to the actual results the next day, given the random factors mentioned (individual variability, incomplete survey information, etc.)?

One approach to this problem is to think of the population's views as having a probability distribution. When a person responds to a survey or votes, it is on the basis of a random sample from this distribution. The problem is then to establish the stability of a solution concept under these conditions.

In the language of probability and statistics, a finite sample of n points from a probability distribution μ corresponds to an empirical measure μ_n. This measure puts mass $1/n$ at each of the n sample points. A solution concept is a function f operating on probability measures, mapping to sets in $\Re^d$. If we could establish that

$$\lim_{n\to\infty} f(\mu_n) \to f(\mu) \ a.e.,$$

then the sample statistic $f(\mu_n)$ would be an asymptotically consistent estimator for $f(\mu)$, and we would be confident of the limiting behavior of the sample solution concept.

Thus the second category of results consists of proofs of consistency of sample estimators for various solution concepts. The new results include the consistency of the sample win set under supermajority voting, the yolk center, and the epsilon-core and critical level.

2 Recovery of spatial locations

In this section we address the problem of "recovering" spatial locations from voting data. From a different point of view, we can define the problem as that of *fitting* the spatial model to data. The idea is, we have data such as roll call votes on how each member of a population voted on a set of issues. We want to determine locations for the voter ideal points, or issue locations, or both, that best fit the data.

Let us be precise. In the recovery problem, we are given the following data: a list of voters indexed $1,\ldots,n$; a list of proposals $1,\ldots,m$; a dimension d; voting data in the form of an n by m matrix A, where

- $A_{ij} = 1$ if voter i voted for proposal j
- $A_{ij} = -1$ if voter i voted against proposal j
- $A_{ij} = 0$ if voter i did not vote on j or the information is not available.

The recovery problem is to specify locations in $\Re^d$ for the voter ideal points and for the "yea" and "nay" of the proposals, that best fit the data A_{ij} according to some criterion. The simplest criterion would be to minimize the number of "errors," *i.e.*, conflicts between the data and the model predictions. When this criterion is used the problem will be referred to as the *simple recovery problem.* More sophisticated weighted error measures are also possible. For example, one could base the measure on a probabilistic model of individual choice as in [9,8], (see [22,21]).

Some researchers locate ideal points by an analysis of the content of the proposals [16, e.g.]. This method has both potential advantages (use of expert knowledge, predictive power) and drawbacks (judgement-based, not easily replicable). The recovery problem defined here may not arise when this method is used. A constrained version of the recovery problem however might arise, if proposal content were used to locate the proposals, and then voting data were used to recover the voter ideal points. This constrained version can be solved optimally by methods presented later in this section. (The content-based method also seems related to the recovery problem in that the study of good solutions to the recovery problem could help clarify the understanding of content, see [23] for example.)

The principal results of this section are:

- It is NP-complete (i.e. computationally difficult in the worst case) to solve the simple recovery problem, even in one dimension.
- In low dimension (e.g. 3 or less) there are computationally efficient algorithms to optimally recover ideal points given fixed proposal locations, and vice-versa (recover proposal locations given fixed ideal points), for the simple recovery problem. Improved heuristic algorithms can then be developed for both simple and general recovery in low dimension.

The complexity result provides theoretical justification for employing heuristic methods to solve the recovery problem, such as the alternating

heuristic of Poole and Rosenthal [22]. The algorithms here lead to some refinements of this heuristic.

2.1 Complexity of the simple recovery problem

Surprisingly, the simple recovery problem is not simple, even in one dimension.

Theorem 1: *The simple recovery problem is NP-complete in one dimension. If the orientation of each proposal is fixed, the problem remains NP-complete.*

Proof: The reduction is from feedback arc set. Given an arbitrary directed graph G, insert a vertex into each arc. Each arc becomes a path of length 2 and the modified graph $\tilde{G}$ is bipartite. Associate a voter with each node in the first part of $\tilde{G}$ and a proposal with each node in the second part. An arc from a voter i to a proposal j will mean a yes vote, $A_{ij} = 1$; an arc from proposal j to voter i will mean a no vote, $A_{ij} = -1$; otherwise $A_{ij} = 0$. Pad the voter set with K "yes" voters, who vote in favor of every proposal, and an equal number of "no" voters, who vote against every proposal. For large K these extra voters force the orientation of each proposal. In particular, K is large enough if $K =$ the number of arcs in $\tilde{G}$. Without loss suppose the yes voters will be placed at the bottom of the ordering and the no voters will be at the top. (This padding ensures that the problem remains hard even if the proposal orientations are fixed.) Then arranging the voter ideal points and proposal centers to minimize the number of errors is equivalent to arranging the vertices of $\tilde{G}$ to minimize the number of downward pointing arcs. This is the feedback arc set problem on $\tilde{G}$.

The last step of the proof is to show this is also the feedback arc set problem on G. Now, minimizing the number of downward pointing arcs is equivalent to determining a minimum cardinality subset $\tilde{S}$ of arcs in $\tilde{G}$ such that every directed cycle in $\tilde{G}$ contains at least one member of $\tilde{S}$. Similarly let S denote the minimum feedback arc set of graph G. Suppose arc (u, v) in G corresponds to arcs $(u, j), (j, v)$ in $\tilde{G}$, *i.e.*, j was the vertex inserted into arc (u, v). The two arcs (u, j) and (j, v) intersect the same cycles in $\tilde{G}$, thus the minimum cardinality $\tilde{S}$ does not contain both. Hence there exists a 1-1 correspondence between members of $\tilde{S}$ and members of S, and minimizing $|\tilde{S}|$ minimize $|S|$ as well.

Finally we observe that the reduction is legitimate, because the matrix A has size polynomial in $|G|$. □

The reduction shows that the recovery problem is similar to the problem of determining the Kemeny consensus, and is hard for related reasons [2]. The key similarity is this: in both cases we are attempting to summarize highly multidimensional information from many voters into a single permutation, and we measure the quality of the summary by counting the disagreements between it and the original multivoter data. Intuitively, this is why both NP-completeness reductions are from the feedback arc set problem, where arcs pointing the wrong way are the disagreements. Of course all NP-complete problems are equivalent in a formal sense; the similarity described here is meant to be illuminating, though informal.

2.2 Optimal recovery of ideal points with respect to fixed proposals

Suppose the proposal locations are fixed. This could arise if they are exogenously determined, or during a step of a heuristic procedure for the recovery problem as in [21]. The recovery problem then reduces to n independent voter location problems. Conversely, if ideal points are fixed, the proposal location recovery problem reduces to m independent location problems. We note that proposal location in the dual or polar space is identical to voter location in the usual space. Thus all results in this subsection apply to both problems.

Theorem 2: *The simple recovery problem with fixed proposals is NP-complete in arbitrary dimension, and polynomially solvable in any fixed dimension. The corresponding proposal location recovery problem with fixed ideal points has the same complexity.*

Proof: Each proposal location can be taken as a hyperplane in $\Re^d$ with orientation (indicating the "yes" halfspace). If a voter is placed in the "wrong" halfspace, an "error" results. The recovery problem therefore is equivalent to the following: given a collection of linear inequalities, find a location in $\Re^d$ that minimizes the number of violated inequalities. This is NP-complete [11, page 267].

In fixed dimension, a well known combinatorial formula (see [3, e.g.]) states that m hyperplanes partition $\Re^d$ into $\sum_{i=0}^{d} \binom{m}{i}$ distinct regions (or

fewer if they are not in general position). Thus the voter must be placed in one of $O(m^d)$ possible regions. Obviously the error count is the same within any region. The region with the least error count is the optimal solution, and can be found in polynomial time. □

The Poole-Rosenthal recovery heuristic selects the coordinates of the locations one dimension at a time. For each dimension, the heuristic alternates between fixing the ideal points and fixing the proposal locations, until a locally optimal solution is reached (the method ceases to improve the solution) [22]. When the coordinates for a dimension have been selected in this way, the heuristic seeks a locally optimal solution for the next dimension, and so on until all are selected [21,23]. When the simple error count is to be minimized, or the random factor is small,Theorem 2 suggests an alternate version of this heuristic, in which all dimensions are considered simultaneously. The more powerful neighborhood structure provided by Theorem 2 suggests that the alternate version would perform better. Of course, given Theorem 1, it is unlikely that the alternate version will *always* dominate the other in numerical performance. To be sure of dominance, one could apply the alternate version to the solution generated by the original heuristic: this hybrid heuristic would be guaranteed always to perform at least as well as the original heuristic.

Theorem 2 hold promise for weighted recovery problems, as well. In the unweighted recovery problem, the penalty is constant within each polyhedral region, so one simply has to pick the best region. In the more general weighted recovery problem, the penalty will vary within each region. If the penalty function behaves tractably, on one side of the proposal hyperplane (e.g. it is convex), then one simply has to solve a convex minimization problem for each region (the sum of convex functions is convex). This would be computationally feasible in two or three dimensions.

Unfortunately, the most widely used penalty functions are not convex. Rather, they tend to flatten out far away from the proposal hyperplane. That is, the penalty for being on the wrong side of the hyperplane is nearly the same at distance 100 or 101; similarly it isn't much better to be on the right side at distance 101 than to be on the right side at distance 100. But there is a big difference between being distance 0 (i.e. on the hyperplane) and distance 1.

For these penalty functions we modify the construction of regions given in the proof of Theorem 2. See Figure 2.2: the penalty function is convex on

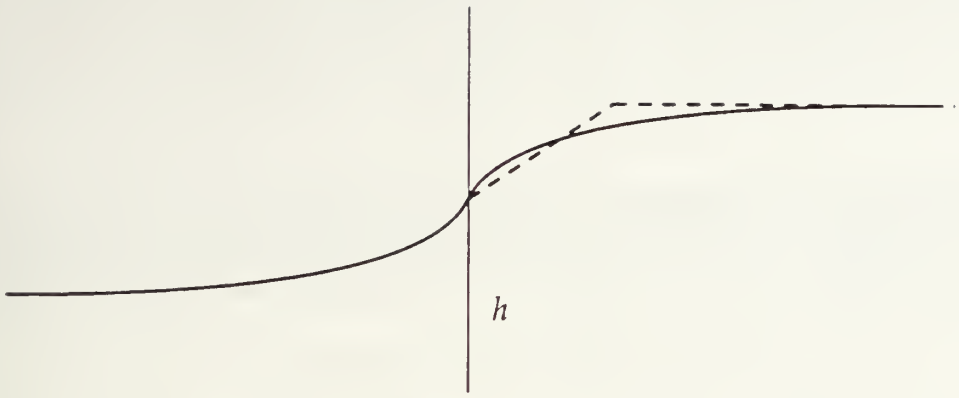

Figure 1: Piece-wise linear approximation of non-convex penalty function

the correct side of the hyperplane, but not on the wrong side. Approximate the penalty on the wrong side by a piecewise linear function. (Note that the piecewise function is concave.) For simplicity suppose there is only one breakpoint. Now place a hyperplane parallel to the proposal hyperplane, at distance corresponding to the breakpoint. If this is done for each proposal, there will be $2m$ hyperplanes in $\Re^d$, and still only $O(m^d)$ regions. (The upper bound will increase by roughly 2^d, a modest amount for $d = 2$ or 3. Actually for $d = 2$ the number of regions increases by a little less than a factor of 3.) Within each of these regions, the penalty function will be a sum of linear and convex functions. Now one simply solves the convex program for each region. If linear programs are desired, approximate the penalty function on the correct side by a piecewise linear function as well. If a better fit is desired, one can use piecewise convex instead of piecewise linear functions, or one can increase the number of breakpoints.

3 Computational methods for Simpson-Kramer win sets

In this and the next two sections, we consider algorithmic procedures for computing solution concepts and associated values. The first solution concept we address involves supermajority or α-majority voting.

Fix the voter ideal points V. For any $1/2 \leq \alpha \leq 1$ let $W(\alpha)$ denote the win set with respect to supermajority rule at level α. Let $\alpha^*(x)$ denote the smallest supermajority level at which x is undominated (in the core), $\inf_\alpha\{\alpha : x \in W(\alpha)\}$. Let $\alpha^*(V)$ denote $\min_x \alpha^*(x)$, the smallest level

at which the core is nonempty. Thus $W(\alpha^*(V))$ is the Simpson-Kramer minimax point. These supermajority win sets have very interesting and powerful properties, see [17,12,5,4]. In this section we develop computational methods for these concepts.

There are several related algorithmic problems:

1. *Membership*: given V, x, and α, is $x \in W(\alpha)$? (Is x undominated with respect to α-majority rule?)

2. *Critical Level*: given V, x, find $\alpha^*(x)$. (What is the smallest supermajority level α at which x is undominated?)

3. *Win Set*: given V, α, find the set $W(\alpha)$ or determine that it is empty. That is, find some "good" representation of the core for the given value of α.

4. *Minimax Level & Point*: given V, find $\alpha^*(V)$ and $W(\alpha^*(V))$. (Find the Simpson-Kramer minimax point and its associated α level.)

As stated in the introduction, all these problems are NP-hard in arbitrary dimension (this follows from the co-NP-completeness of determining if $x \in W(1/2)$ [1,14]). Fortunately, all these problems may be solved fairly easily in low dimension.

Algorithms for (1)-(4)

(1): Membership. The point x is undominated with respect to α-majority rule iff no open halfspace defined by a hyperplane through x contains more than $\alpha|V|$ ideal points. Therefore we can determine membership by finding the "densest" open halfspace, the open halfspace containing the most points of V, and counting the number of points it contains. Johnson and Preparata [14] provide polynomial algorithms to find the "densest" closed and open halfspaces with respect to a set of points in fixed dimension. This resolves membership in fixed dimension.

One drawback to the algorithm of [14] is that it is moderately complicated to implement. Here we suggest an alternate algorithm that is conceptually simple, and has the same order of speed on nondegenerate cases. The algorithm should be easy to implement, particularly as it requires computations similar to those for the yolk. The principal disadvantage to our

algorithm is its relatively poor performance on degenerate configurations, although it is still "fast" (polynomial time) for any fixed dimension. In four or more dimensions, the algorithm of [14] would be much faster on highly degenerate cases.

Construct all the "extremal" or "limiting" hyperplanes through x, and count the number of ideal points in each halfspace. (A limiting hyperplane passes through d affinedly independent ideal points in $\Re^d$. The point x is counted as an ideal point, whether or not $x \in V$.) This finds the densest closed halfspace, with the same time complexity as [14]. The search for a densest closed halfspace be restricted to just the "limiting" hyperplanes, even for degenerate configurations, because any hyperplane through x can be "tilted" so as to touch d ideal points, while keeping at least as many ideal points in each closed halfspace defined by the hyperplane [25, Lemma 5]. Let h denote the hyperplane defining this halfspace, and let V_h denote the points of V on h, excluding x. If the configuration is nondegenerate, it is possible to perturb h to a new hyperplane $\tilde{h}$, which passes through x, and leaves all the points V_h on its "dense" side. This perturbation is possible because $|V_h| = d - 1$ when the configuration is nondegenerate.

Note that the fundamental operations involved are: (i) finding the limiting hyperplanes, and (ii) counting the number of points in the halfspaces so defined. These operations comprise part of the fundamental operations in yolk computations, which should be an advantage to implementors. (In addition, one must count the number of points on h to verify $|V_h| = d - 1$, but this is trivial given (ii). In two (or more) dimensions, the algorithm can be sped up by a factor of $n/\log n$ by the preprocessing of V [7], just as with yolk computations [31]. The algorithm requires time $O(n^{d-1} \log n)$ in the nondegenerate case (or for the general case, if a densest closed halfspace is desired), the same as the algorithm of [14].

It remains to modify the algorithm in the case of degeneracy. If configurations are degenerate, it may not be possible to perturb h to get all the points of V_h on the same side. (Indeed the densest open halfspace might not be a perturbation of the densest closed halfspace. However, the densest open halfspace is defined by a perturbation of some limiting hyperplane.) Fortunately, the problem of finding the best perturbation of h, that which gets as many points as possible on the dense side, is precisely the densest hemisphere problem in a lower dimensional space. This simplifies implementation considerably.

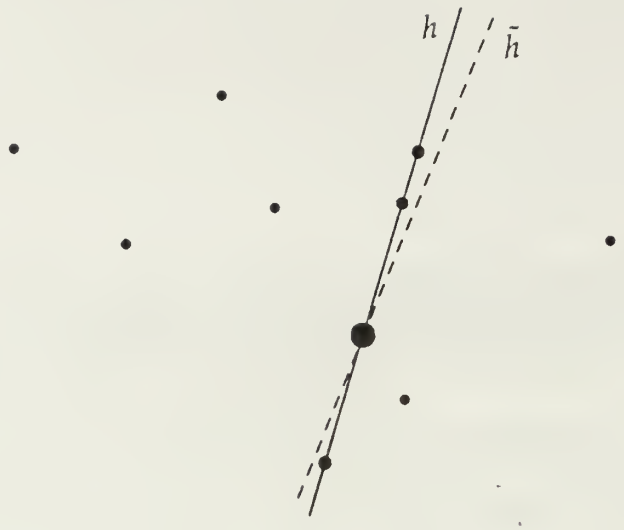

Figure 2: Finding $\tilde{h}$ in 2 dimensions

To be precise, we must find all limiting hyperplanes h, and for each h find its best perturbation $\tilde{h}$. If $|V_h| = d - 1$ this requires no additional computation. If there are more points on h, we recursively solve the densest open hemisphere problem at x for the point set V_h, in the $d-1$ dimensional subspace h. The base case of the recursion is two dimensions: when $d = 2$, (see Figure 3) h is a line through x passing through 2 or more (other) points of V. This line is divided into two half lines emanating from x. Select the half line with the greater number of points. □

Both algorithms extend immediately to the case of weighted voters.

(2): Critical Level. This could be accomplished with binary search on α using (1). However, examination of either algorithm for (1) shows that they actually compute $\alpha^*(x)$ directly, since they find the densest open halfspace. □

(3): Win Set. Find all the "limiting" hyperplanes h (those passing through d ideal points). Form a list E of all closed halfspaces defined by these h. Extract a list L of closed halfspaces h^+ containing $\alpha n + 1$ or more ideal points. If a point is to be in $W(\alpha)$ it must be in each of these closed halfspaces, for otherwise some other point could get $\alpha n + 1$ or more supporters to defeat it. Therefore the win set is the set of feasible solutions to the linear program that constrains points to be in each closed halfspace on the list L. □

As a corollary, we find that the win set is a polyhedral set. (This was established for the special case $\alpha = \alpha^*$ by Kramer [17]).

(4): Minimax Level. This can be found, in principle, by binary search with (3). A direct implementation would proceed as follows: sort the list

E from (3) (there are $O(n^d)$ in this list) in order from largest to smallest number of ideal points contained (descending order of $|V \cap h^+|$). Perform binary search on the cutoff point of E. Each "query" for the binary search solves a feasibility linear program with $O(n^d)$ constraints in d dimensions. (For computational purposes, the dual would probably be faster to solve, since there are far more constraints than variables in the primal.) □

If preferences are linear rather than Euclidean (see [4]) all these problems reduce to (2), Critical level. This is because we need to know the densest halfspace, the open halfspace containing as many preference gradients as possible. As already observed, the densest halfspace problem is solved in [14] or may be solved with the alternative algorithm described above for (1).

4 Computational methods for the yolk

The yolk is another important solution concept, motivated by experimental data and possessed of beautiful theoretical properties [10,18]. As with the Simpson-Kramer minimax point, it is NP-hard to determine the yolk in arbitrary dimension (any solution concept which coincides with the classical core, when the core is nonempty, will suffer from this complexity because it is co-NP-complete to determine if the classical core is nonempty [1]).

For two dimensions, the computational situation is much rosier. Although the extremal median hyperplanes employed in Section 3 are not enough to determine the yolk [27], there is a polynomial algorithm to compute the yolk radius and center, in fixed dimension [31]. In two dimensions, this algorithm requires provably $O(n^{4.5})$ time, and has even better time complexity $o(n^{3+\epsilon})$ $\forall \epsilon > 0$ (i.e. nearly cubic) if a conjecture of Erdös, Lovasz, *et al.* is true.

We should also remark that the linear program given in [18] is easily computed and should make an excellent heuristic. It provides a rigorous lower bound on the yolk radius, and when its solution is exact this event can be determined in time $O(n^2)$ [31].

Koehler has proposed extending the yolk to n even. The algorithm cited above also works in this case, with the same time complexity. The algorithm also works for "supermajority rule" yolks (see [16,28]).

For the 3-dimensional case, the algorithm of [31] has time complexity $O(n^{10})$ and would not be practical for more than a handful of points. This case needs more development but appears to be within reach through either effective heuristics or an improved optimizing procedure, perhaps making strong use of the convexity of the yolk radius function.

5 Computational methods for other solution concepts

The uncovered set

The uncovered set is another solution concept with powerful theoretical properties. It is defined as follows: alternative x *covers* alternative y iff x defeats (is majority preferred to) y, and for all z which y defeats, x defeats z. An alternative x is *uncovered* iff there is no other alternative that covers x [20,19].

The computational picture with respect to the uncovered set is bleak at present. Hartley and Kilgour [13] show how to find the uncovered set in two dimensions when $n = 3$, and their method is not simple. To the author's knowledge, no efficient general procedure is available even for two dimensions, although no forbidding complexity levels (e.g. NP-completeness) have been established for the fixed dimensional case, either. This remains a challenging open problem.

To get an idea of why the problem is not simple, consider the presumably easier question of determining whether a given alternative x covers another given alternative y, in two dimensions. For any point $z \in \Re^2$ and any $0 \leq \theta \leq 2\pi$, let $f(z,\theta) \geq 0$ denote the distance from z along the ray at angle θ to the first point that z defeats. (Technically f is defined as an infimum to allow $f = 0$). It is obvious that z will defeat all points on the ray past this first point. It is fairly straightforward to see that for any z, $f(z,\theta)$ takes the form of a piecewise trigonometric function with polynomially many breakpoints. Then the question of whether x covers y is essentially the question of whether the exterior region in the plane of one such function about x contains the exterior region of another such function about y. This could be checked in polynomial time, because each pair of the trigonometric functions will intersect a small number of times, but the

procedure would be tedious to say the least.

Therefore, to determine whether x covers y is polynomially solvable, but seems near the limit of computational practicality. Fast methods for determining the uncovered set in 2-D may be difficult to attain. On the brighter side, McKelvey [18] provides bounds on the uncovered set in terms of the yolk, which can be computed by the algorithm cited in Section 4.

The epsilon-core

Another solution concept of interest is the epsilon-core: a point x is considered undominated if there is no point y which is more than ϵ closer to a majority of the voters. This concept was introduced by Shapley and Shubick [26]; promising empirical results are given in [24]; some motivations and properties of the epsilon core for the spatial model are given in [30].

The epsilon-core presents the same four algorithmic problems as super-majority voting. These are,

1. *membership:* given ϵ, x is x in the ϵ-core?
2. *critical level:* for given x, find the least value of ϵ for which x is undominated.
3. *win set:* for given ϵ, find the epsilon-core.
4. *minimum level:* find the least value of ϵ for which the epsilon-core is nonempty.

Here we give here fast computational methods to solve the first two problems, *membership* and *critical level*, in 2-D. At present I do not know computationally efficient methods to solve the other problems, *win set* and *minimum level.*

Algorithms for epsilon-core membership and critical level

Membership: For given ϵ, the point x is in the ϵ-core if none of the hyperbolic regions, with one focus at x and parameter ϵ, contain more than half the voter ideal points. These regions are all congruent and are found by rotating a hyperbolic curve around x. The number of points in the region inside the curve only changes when the curve passes over an ideal point. Therefore we can restrict our attention to the $2(n-1) = O(n)$ curves that intersect an ideal point. If none of these contain more than half the voter ideal points,

then x is undominated. The *membership* algorithm is therefore $O(n^2)$. (The same idea works in higher (fixed) dimensions, except one must be careful if the ideal points are sparse: when n is much larger than d the method extends easily.) □

Critical level: This is solved by binary search on ϵ, using the membership algorithm as the query subprocedure. □

As an approximate method for win set and minimum level, the critical level algorithm could be invoked for each point of a grid. Decreasing the spacing of the grid would give as fine an approximation as desired.

6 Convergence and asymptotic consistency of sample estimators

In this section we survey convergence results for several solution concepts, as motivated in the introduction.

6.1 Simpson-Kramer win sets

There are several related convergence questions here, just as there are several related algorithmic questions in Section 3. Corresponding to "Membership" we could define a 0-1 valued random variable which indicates whether x is an α-majority winner or not. Unfortunately it is not the case that the sample statistic would be consistent. In fact we could have the sample statistic converge to 0 *a.s.*, while the distribution value equals 1 (see [29]). The other convergence questions have nicer answers.

- *Minimax value:* Does $\alpha^*(\mu_n) \to \alpha^*(\mu)$?
- *Minimax set:* Does $S_{\alpha^*(\mu_n)}(\mu_n) \to S_{\alpha^*(\mu)}(\mu)$?
- *Critical level:* Does $\alpha(x, \mu_n) \to \alpha(x, \mu)$?
- *Win set:* Does $S(\alpha, \mu_n) \to S(\alpha, \mu)$ in some suitable topology?

Demange [6] considers a sequence of probability measures μ_n converging to μ and shows convergence of Minimax value and Minimax set. The proof appears to assume continuity of μ_n, and may not apply to the case considered

here, where μ_n is a discrete (empirical) measure. Caplin and Nalebuf [5] establish *a.s.* convergence of Minimax value under stated assumptions that μ is concave and has compact support. As they comment, however, their proof applies equally well to the more general case of continuous densities μ with compact support. Moreover, their proof also shows convergence of Critical level. Convergence of Minimax value for arbitrary μ is proved in [29].

Minimax set and Win set are a little different and appear to require stronger assumptions on μ. These questions ask if the location of the solution set converges, rather than if the necessary parameter level for the nonemptiness of the solution set converges. In [29], convergence of Minimax set is proved, under the assumption that μ is continuous and has unique minimax point. Convergence of the Win set will follow from arguments given here, under the assumption that μ is continuous and has compact support (see Section 6.3).

6.2 The yolk

There are two related convergence questions for the yolk, pertaining to its size and location. Let $r(\mu)$ and $c(\mu)$ denote the radius and center, respectively, of the yolk with respect to a distribution μ. Convergence of $r(\mu_n)$ to $r(\mu)$ $(w.p.1)$ is proved in [30], assuming μ is continuous and strictly positive in its region of support. If no restrictions are placed on μ, the sample radius $r(\mu_n)$ may not have a limit. I conjecture, moreover, that in sufficiently high dimension, it can occur that $\limsup_n r(\mu_n) < r(\mu)$.

Here we prove the convergence of the yolk center.

Theorem 4: If μ is positive continuous on $\Re^d$ and has compact support then $\lim_{n\to\infty} c(\mu_n) \to c(\mu)$ *a.s.*

Proof: We will need two lemmata.

Lemma 1: Let $r(z,\mu)$ denote the radius of the z-centered yolk. Then $lim_{n\to\infty} r(z,\mu_n) \to r(z,\mu)$ *a.s.* uniformly over all z.

(This is proved in [30].)

Lemma 2: The function $r(z,\mu)$ is continuous and convex in z.

(The proof is easy and found in [31]).

Let c denote $c(\mu)$, the center of the distributional yolk. Then $r(c,\mu_n) \to r(\mu)$ *a.s.* by Lemma 1. Now let $\epsilon > 0$ be arbitrary. We will show that for

all $z \in \Re^d$ more than distance ϵ from c, eventually $z \notin c(\mu_n)$ w.p. 1. This will imply that $c(\mu_n)$ must almost surely converge to points within c.

Let D denote the set of points in the support of μ, at distance ϵ or more from c. D is compact so by Lemma 2, $r(z,\mu)$ attains a minimum value r_D when z ranges over D. By assumption D does not intersect c. Hence $r_D > r(\mu)$.

Let $\delta = r_D - r(\mu) > 0$. By Lemma 1, eventually as $n \to \infty$

$$\sup_z |r(z,\mu_n) - r(z,\mu)| < \delta/3 \quad a.s.$$

So eventually for all $z \in D$ and all points p in c we have, a.s., $r(z,\mu_n) \geq r_D - \delta/3 > r_D - 2\delta/3 \geq r(p,\mu_n)$. Hence eventually $z \notin c(\mu_n)$ $a.s.$ since there is another point p with smaller p-centered radius. □

Lemma 3 in [31] gives a related result: if V is any finite configuration of ideal points, and $\tilde{V}$ is an arbitrarily slight nondegenerate perturbation of V, then the yolk radius and center of $\tilde{V}$ can be made arbitrarily close to the yolk radius and center of V. In the case $|V|$ even, the center of V may not be unique. In this case, the yolk center of $\tilde{V}$ can be made arbitrarily close to some (but not any) yolk center of V. This result, too, guarantees a certain robustness of the yolk solution concept.

6.3 The epsilon-core

In this subsection we show convergence properties related to the epsilon-core. This solution concept and the α-majority concept have similar properties.

Let $\epsilon(x,\mu)$ denote the smallest value of ϵ that makes x undominated; let $\epsilon^*(\mu)$ denote the smallest value of ϵ for which the epsilon-core is nonempty; let $EC(\epsilon,\mu)$ denote the epsilon-core at value ϵ; let $EC^*(\mu)$ denote the epsilon-core at value $\epsilon^*(\mu)$.

Theorem 5: If μ is continuous on $\Re^d$ and has compact support then with probability 1,

1. $\epsilon(x,\mu_n) \to \epsilon(x,\mu)$.
2. $\epsilon^*(\mu_n) \to \epsilon^*(\mu)$.
3. $\forall \epsilon > 0$, $EC(\epsilon,\mu_n) \to EC(\epsilon,\mu)$. In particular, if $\epsilon^*(\mu) > 0$, then $EC^*(\mu_n) \to EC^*(\mu)$.

4. $\forall \alpha,\ S(\alpha, \mu_n) \to S(\alpha, \mu)$.

Proof: The key property we rely on is that the probability measure of the region defined by a hyperbolic hypersurface changes continuously as the surface is translated continuously through $\Re^d$. This follows immediately from the assumptions. Now apply the uniform convergence of the empirical measure μ_n to μ over all these hyperboloid regions ([30, Lemma 2]) to get convergence of the critical level $\epsilon(x, \mu)$ and the minimum level ϵ^*.

To prove convergence of the win sets $EC(\epsilon)$, we use the same argument as for the convergence of the yolk center in the previous subsection. That is, once we know critical level and minimum level converge, the continuity of the probability measure, and the compactness of its support, imply that for all $\eta > 0$ eventually all points more than η from the win set $EC(\epsilon, \mu)$ will have critical level greater than ϵ. The same proof works for convergence of $S(\alpha, \mu_n)$, the minimax win set, as well. □

7 Computations for large data sets and directions for further research

7.1 Computing solution sets for very large populations

If a population is very large, even the relatively fast algorithms of section 2 may be too slow for practical purposes. For these situations[2] we may combine the two categories of results in this article, to get fast, accurate computations. Suppose two random samples of n points are drawn from a distribution μ, leading to empirical measures μ_n and μ'_n. Suppose further we are interested in $f(\mu)$, where the a.s. asymptotic consistency of the sample estimates $f(\mu_n)$ is established. Then since both $f(\mu_n)$ and $f(\mu'_n$ converge to $f(\mu)$ a.s., it must be that $lim_{n\to\infty} f(\mu_n) = lim_{n\to\infty} f(\mu'_n)$ a.s.

Now suppose we have a data for a large population P. The data may be thought of as a very large sample from some continuous distribution μ. By the asymptotic equivalence between sampling with and without replacement, we can approximate a sample from μ by a sample from P. Think of

[2] this idea is due to Richard McKelvey

P as mu_n. To estimate $f(P)$, we draw samples μ'_m from P, and compute $f(\mu'_m)$. The equality of limits in the preceding paragraph assures us that (for large enough m) the sample value $f(\mu'_m)$ is a good estimate of the value of $f(\mu_n) = f(P)$. If n is very large, the required sample size m would be much smaller than n, rendering the problem computationally tractable. (Note, if P has multiplicities, this reasoning fails. I think the method would still work well, but have no mathematical justification, except in the case of the yolk, where Lemma 3 in [31] would fill in the gap.)

7.2 Directions

This paper has assembled a number of computational and convergence results in support of empirical research with the spatial model. As indicated in the text, several computational problems remain open. These include finding good methods for the uncovered set and epsilon-core win sets, and also for the yolk in 3 or more dimensions.

Regarding convergence results, finding *rates* of convergence appears to be an important open research area. If these could be estimated, and the role of the variance of μ clarified, it should then be possible to construct statistical tests (e.g. confidence intervals) for empirically computed values such as the yolk radius and location, or the minimax level.

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